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Uniqueness of limit cycles in theoretical models of certain oscillating chemical reactions

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Abstract
A criterion for the uniqueness of limit cycles for generalized Liénard-type system is established. Through a simple change of variable, three planar autonomous systems, which arise from the study of theoretical models for oscillating reactions, can be transformed into a better studied generalized Liénard-type system. As a result, the uniqueness of limit cycles of these systems is obtained.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the study of the two-dimensional dynamical system

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y),$$

the question of the number of limit cycles is often encountered. Many published works refer to specific classes of systems, for example, the Liénard system

$$x'(t) = \pi(y) - h(x), \qquad y'(t) = -\psi(x), \tag{1.1}$$

and predator–prey system,

$$x'(t) = \varphi(x)(h(x) - \pi(y)), \qquad y'(t) = \rho(y)\psi(x), \tag{1.2}$$

etc. There exists extensive literature on them, not least because they arise in frequent applications. For the uniqueness of limit cycles for (1.1), criteria which have been widely used are those in Cherkas and Zhilevich [1] and Zhang [2]. Based on the requirement of monotonicity of h'/ψ, they compared the Floquet exponent of periodic solutions and

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concluded that the innermost periodic solution will reach the largest Floquet exponent. Then according to the theory of a rotated vector field [3], one can obtain that the innermost periodic solution is orbitally asymptotically stable. Hence, the uniqueness of limit cycles follows. The idea of the proof was modified by other authors (see Ding [4] and Hwang [5]) to establish the uniqueness of limit cycles of (1.2). By using the symmetry of prey isocline, Cheng [6] proved the uniqueness of limit cycles for a specific predator–prey system with Holling type II functional response and the idea was extended by Liou and Cheng [7] to a predator–prey system of Gauss type and obtained some criteria of uniqueness. Hsu and Hwang [8] use the method of reflection to obtain a sufficient condition for the uniqueness of limit cycles of a Leslie-type predator–prey system. By transforming (1.2) into (1.1), Kuang and Freedman [9], and Huang and Merrill [10] obtained the uniqueness theorem of (1.2) with a general functional response. Recently, a criterion of (1.1), without the monotonicity of h'/ψ, was provided by Xiao and Zhang [11].

In the next section, with a slight modification of the theorem obtained by Hwang [5], we establish conditions to ensure that the number of limit cycles of the following system (1.3) does not exceed 1:

$$\begin{aligned} x'(t) &= \varphi(x)(\pi(y) - h(x)) \equiv F(x, y),\\ y'(t) &= -\psi(x) \equiv G(x, y),\\ x(0) &= x_0, \qquad y(0) = y_0, \end{aligned} \tag{1.3}$$

where φ, h, ψ are $\mathcal{C}^1$ on $(r_1, r_2) \subseteq \mathbb{R}$ and π is $\mathcal{C}^1$ on $\mathbb{R}$. Moreover, all these functions satisfy the following assumptions:

(A1) $\pi'(y) > 0$ for $y \in \mathbb{R}$;
(A2) $\varphi(x) > 0$ for $x \in (r_1, r_2)$;
(A3) There exists $\lambda \in (r_1, r_2)$ such that $\psi'(\lambda) > 0$ and $(x-\lambda)\psi(x) > 0$ for $x \in (r_1, r_2) - \{\lambda\}$;
(A4) $h((r_1, r_2)) \subseteq \pi(\mathbb{R})$.

Note that system (1.3) is exactly the Liénard system if $\varphi(x) = 1$ for $x \in (r_1, r_2)$. Hence, our results can be viewed as an extension of those obtained by Cherkas and Zhilevich for (1.1). In section 3, three planar autonomous systems, which arise from the study of theoretical models for oscillating reactions, are provided to show the applicability of the main theorems. Finally, a brief concluding remark is given in section 4.

2. Main results

It is clear that system (1.3) has equilibrium at $e_\star = (\lambda, y_\star)$, where $y_\star = \pi^{-1}(h(\lambda)) > 0$. The Jacobian of system (1.3) at $e_\star$ is

$$J = \begin{bmatrix} -\varphi(\lambda)h'(\lambda) & \varphi(\lambda)\pi'(y_\star) \\ -\psi'(\lambda) & 0 \end{bmatrix}.$$

The eigenvalues are given by

$$\left[-\varphi(\lambda)h'(\lambda) \pm \sqrt{(\varphi(\lambda)h'(\lambda))^2 - 4\varphi(\lambda)\psi'(\lambda)\pi'(y_\star)}\right]\Big/2.$$

Hence $e_\star$ is stable if $h'(\lambda) > 0$ and $e_\star$ is unstable if $h'(\lambda) < 0$.

Theorem 2.1. *Let the assumptions (A1)–(A4) hold. Assume that there exists $a, b \in \mathbb{R}$ such that*

$$0 \neq \varphi(x)h'(x) + a\psi(x) + b\psi(x)h(x) \geqslant 0, \qquad \text{on} \quad (r_1, r_2).$$

Then system (1.3) has no periodic solution in $(r_1, r_2) \times \mathbb{R}$.

Proof. It is sufficient to prove that $\Omega = (r_1, r_2) \times \mathbb{R}$ contains no periodic solution of system (1.3). Let $H(x, y) = l(x)r(y)$, where

$$r(y) = \exp\left(\int_{y_\star}^{y} (a + b\pi(\eta))\, \mathrm{d}\eta\right)$$

and

$$l(x) = (\varphi(x))^{-1} \exp\left(b \int_{\lambda}^{x} \frac{\psi(\xi)}{\varphi(\xi)}\, \mathrm{d}\xi\right).$$

Then,

$$\frac{r'(y)}{r(y)} = a + b\pi(y), \qquad \frac{\varphi'(x)}{\varphi(x)} + \frac{l'(x)}{l(x)} = b\frac{\psi(x)}{\varphi(x)},$$

and

$$\begin{aligned}\triangle &= \nabla \cdot (FH, GH)\\ &= -H(x, y)\bigg[\varphi(x)h'(x) + \varphi(x)h(x)\left(\frac{\varphi'(x)}{\varphi(x)} + \frac{l'(x)}{l(x)}\right)\\ &\quad - \pi(y)\varphi(x)\left(\frac{\varphi'(x)}{\varphi(x)} + \frac{l'(x)}{l(x)}\right) + \psi(x)\frac{r'(y)}{r(y)}\bigg]\\ &= -H(x, y)(\varphi(x)h'(x) + a\psi(x) + b\psi(x)h(x)) \leqslant 0 \qquad \text{on} \quad \Omega.\end{aligned}$$

Hence, the assertion follows by the Dulac criterion. □

Theorem 2.2. *Suppose that $h'(\lambda) < 0$ and (A1)–(A4) hold and, moreover, there exist $\alpha, \beta \geqslant 0$ such that*

(A5) $\alpha + \beta h(x) > 0$ for all $x \in (r_1, r_2)$;
(A6) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha+\beta h(x))}\right) \geqslant 0$ for all $x \in (r_1, r_2) - \{\lambda\}$.

Then system (1.3) possesses at most one limit cycle, and if it exists then it is stable.

Proof. If $\beta = 0$ then, from (A5), we obtain $\alpha > 0$. Hence, the assumption (A6) is exactly the condition provided by Zhang [2]. Thus, we only consider $\beta > 0$. Without loss of generality, we may assume that system (1.3) has nontrivial periodic orbits. Let $\Gamma(t) = (x(t), y(t))$ be any periodic solution of (1.3) and $a, b \in \mathbb{R}$; one obtains

$$\psi(x(t))(a + bh(x(t))) = -(a + b\pi(y(t)))y'(t) - b\frac{\psi(x(t))}{\varphi(x(t))}x'(t)$$

and

$$\nabla \cdot (F, G)(\Gamma(t)) = -\varphi(x(t))h'(x(t)).$$

Consequently,

$$\oint_\Gamma \nabla \cdot (F, G)(\Gamma(t))\, \mathrm{d}t = \oint_\Gamma [-\varphi(x)h'(x) - \psi(x)(a + bh(x))]\, \mathrm{d}t.$$

Since $e_\star$ is unstable, there must be a periodic orbit Γ_1 which is the nearest one around $e_\star$. It follows that Γ_1 must be stable from inside, and by the Poincaré criterion of stability, we get

$$\oint_\Gamma \nabla \cdot (F, G)(\Gamma(t))\, \mathrm{d}t \leqslant 0.$$

Let $x_1 = \min\{x | (x, y) \in \Gamma_1\}$ and $z = \max\{x | (x, y) \in \Gamma_1\}$. Define

$$a = -\frac{\varphi(x_1)h'(x_1)}{\psi(x_1)(\alpha + \beta h(x_1))}\alpha, \qquad b = -\frac{\varphi(x_1)h'(x_1)}{\psi(x_1)(\alpha + \beta h(x_1))}\beta, \tag{2.1}$$

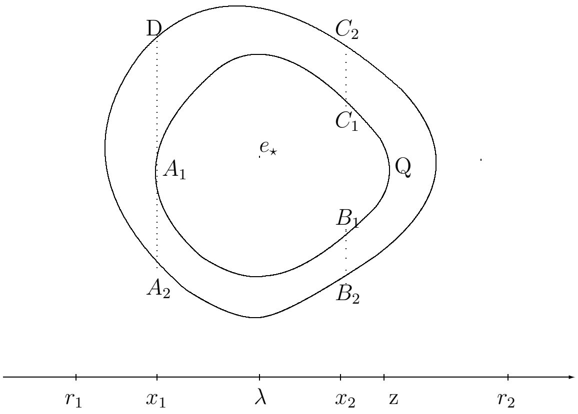

Figure 1.

and

$$w(x) = \varphi(x)h'(x) + a\psi(x) + b\psi(x)h(x).$$

Clearly, $x_1 \in (r_1, \lambda)$ and $w(x_1) = 0$. Since $w(x)/\psi(x)[\alpha+\beta h(x)]$ is non-decreasing in (r_1, r_2) and $\psi(x) < 0$ in (r_1, λ), we have $w(x) > 0$ as $x \in (r_1, x_1)$ and $w(x) < 0$ as $x \in (x_1, \lambda)$. If $w(x) \leqslant 0$ as $x \in (\lambda, z)$ then

$$0 \geqslant \oint_{\Gamma_1} \nabla \cdot (F, G)\,\mathrm{d}t = -\oint_{\Gamma_1} w(x(t))\,\mathrm{d}t > 0,$$

a contradiction. Hence, there must exist an x_2 such that $w(x) < 0$ as $x \in (x_1, x_2)$ and $w(x) > 0$ as $x \in (r_1, x_1) \cup (x_2, r_2)$. Suppose there exists another periodic orbit Γ_2 that is outside and closest to Γ_1. The vertical line $x = x_1$ intersects the orbit Γ_2 at points A_2 and D (see figure 1). The vertical line $x = x_2$ intersects Γ_1 and Γ_2 at points B_1, C_1 and B_2, C_2, respectively. Then,

$$\oint_{\Gamma_1} w(x)\,\mathrm{d}t = \left(\int_{\widehat{A_1C_1}} + \int_{\widehat{C_1B_1}} + \int_{\widehat{B_1A_1}}\right) w(x)\,\mathrm{d}t,$$

$$\oint_{\Gamma_2} w(x)\,\mathrm{d}t = \left(\int_{\widehat{A_2D}} + \int_{\widehat{DC_2}} + \int_{\widehat{C_2B_2}} + \int_{\widehat{B_2A_2}}\right) w(x)\,\mathrm{d}t.$$

Let $y = y_1(x)$ and $y = y_2(x)$ denote the functions of curves $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, respectively. Then,

$$\begin{aligned}
&\int_{\widehat{B_2A_2}} w(x)\,\mathrm{d}t - \int_{\widehat{B_1A_1}} w(x)\,\mathrm{d}t \\
&\quad = \int_{x_2}^{x_1} \frac{w(x)}{\varphi(x)(\pi(y_2(x)) - h(x))}\,\mathrm{d}x - \int_{x_2}^{x_1} \frac{w(x)}{\varphi(x)(\pi(y_1(x)) - h(x))}\,\mathrm{d}x \\
&\quad = -\int_{x_1}^{x_2} \frac{w(x)}{\varphi(x)} \frac{\pi(y_1(x)) - \pi(y_2(x))}{[\pi(y_1(x)) - h(x)][\pi(y_2(x)) - h(x)]}\,\mathrm{d}x > 0.
\end{aligned}$$

Similarly, we can prove

$$\int_{\widehat{DC_2}} w(x)\,\mathrm{d}t - \int_{\widehat{AC_1}} w(x)\,\mathrm{d}t > 0.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha+\beta h(x))}\right)>0$$

as $x\in(r_1,r_2)-\{\lambda\}$ and $w(x_1)=w(x_2)=0$, one has

$$\begin{aligned}\int_{\widehat{A_2D}} w(x)\,\mathrm{d}t &= \left(\int_{\widehat{A_2D}}+\int_{\overline{DA_2}}\right) w(x)\,\mathrm{d}t\\ &=\oint_{A_2DA_1A_2}\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\left[-\beta\frac{\psi(x)}{\varphi(x)}\,\mathrm{d}x-(\alpha+\beta\pi(y))\,\mathrm{d}y\right]\\ &=\oint_{A_2A_1DA_2}\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\left[\beta\frac{\psi(x)}{\varphi(x)}\,\mathrm{d}x+(\alpha+\beta\pi(y))\,\mathrm{d}y\right]\\ &=\int\!\!\int_{\Omega_1}(\alpha+\beta\pi(y))\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\right)\mathrm{d}x\,\mathrm{d}y>0\end{aligned}$$

where $\overline{DA_2}$ is line segment from point D to point A_2, and

$$\begin{aligned}&\int_{\widehat{C_2B_2}} w(x)\,\mathrm{d}t-\int_{\widehat{C_1B_1}} w(x)\,\mathrm{d}t\\ &\quad=\int_{B_2B_1\widehat{QC_1}C_2B_2}\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\left[-\beta\frac{\psi(x)}{\varphi(x)}\,\mathrm{d}x-(\alpha+\beta\pi(y))\,\mathrm{d}y\right]\\ &\quad=\int_{B_2C_2\widehat{C_1Q}B_1B_2}\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\left[\beta\frac{\psi(x)}{\varphi(x)}\,\mathrm{d}x+(\alpha+\beta\pi(y))\,\mathrm{d}y\right]\\ &\quad=\int\!\!\int_{\Omega_2}(\alpha+\beta\pi(y))\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w(x)}{\psi(x)(\alpha+\beta h(x))}\right)\mathrm{d}x\,\mathrm{d}y>0\end{aligned}$$

where Ω_1 and Ω_2 are two regions bounded by the above two closed paths, respectively. Thus,

$$\oint_{\Gamma_2}\nabla\cdot(F,G)\,\mathrm{d}t=-\oint_{\Gamma_2}w(x)\,\mathrm{d}t<-\oint_{\Gamma_1}w(x)\,\mathrm{d}t=\oint_{\Gamma_1}\nabla\cdot(F,G)\,\mathrm{d}t\leqslant 0.$$

Since two periodic orbits with the same stability cannot exist side by side, we conclude that Γ_1 is externally unstable. To obtain a contradiction, let $\theta(x)$ and $H(x)$ be the solutions of the following initial value problems:

$$\theta'(x)=\frac{\beta w(x)}{\alpha+\beta h(x)}\frac{\theta(x)}{\varphi(x)}(1+\theta(x)),\qquad \theta(x_2)=1;\tag{2.2}$$

and

$$H'(x)+b\frac{\psi(x)}{\varphi(x)}H(x)=\frac{\theta(x)}{\varphi(x)}w(x),\qquad H(x_2)=0,\tag{2.3}$$

respectively. Note that $\theta(x)>0,\theta'(x)>0$ for $x\in(x_2,r_2)$ and they are defined. Moreover, since $\left(\exp\left(b\int_{x_2}^{x}\frac{\psi(\xi)}{\varphi(\xi)}\,\mathrm{d}\xi\right)H(x)\right)'=\exp\left(b\int_{x_2}^{x}\frac{\psi(\xi)}{\varphi(\xi)}\,\mathrm{d}\xi\right)\frac{\theta(x)}{\varphi(x)}w(x)>0$ as long as $\theta(x)$ is defined, one has $H(x)>0$ as long as $\theta(x)$ is defined. Now consider the new system

$$\begin{aligned}x'(t)&=\varphi(x)(\pi(y)-h_\varepsilon(x))\equiv F_\varepsilon(x,y),\qquad \varepsilon\geqslant 0\\ y'(t)&=-\psi(x)\end{aligned}\tag{2.4}$$

where

$$h_\varepsilon(x)=\begin{cases}h(x) & \text{if}\quad x\in(r_1,x_2]\\ h(x)+\varepsilon H(x) & \text{if}\quad x\in(x_2,r_2).\end{cases}$$

Clearly, $h_\varepsilon\in C^1$ for $\varepsilon\geqslant 0$, and system (2.4) satisfies assumptions (A1)–(A5). Systems (2.4) and (1.3) are identical on $(r_1,x_2]$ and (2.4) forms a family of rotated vector fields with respect to ε on $x \in (x_2, r_2)$, and hence forms a family of generalized rotated vector fields on Ω. As $0 < \varepsilon \ll 1$ the semi-stable limit cycle Γ_1 will split into at least two limit cycles Γ_1' and Γ_1'', where Γ_1' is enclosed by Γ_1'' and, moreover, Γ_1'' is at least unstable on the outside and Γ_1' is at least stable from the inside, i.e.,

$$\oint_{\Gamma_1''} \nabla \cdot (F, G)\,\mathrm{d}t \geqslant 0 \geqslant \oint_{\Gamma_1'} \nabla \cdot (F, G)\,\mathrm{d}t.$$

If we can show that the assumption (A6) holds for system (2.4), then by applying a similar argument at the beginning of the proof, one obtains

$$\oint_{\Gamma_1''} \nabla \cdot (F_\varepsilon, G)\,\mathrm{d}t < \oint_{\Gamma_1'} \nabla \cdot (F_\varepsilon, G)\,\mathrm{d}t,$$

a contradiction. So, the proof will be completed if the assumption (A6) holds for system (2.4). To see this, let

$$\Delta_\varepsilon(x) = \frac{\varphi(x)h_\varepsilon'(x)}{\psi(x)[\alpha + \beta h_\varepsilon(x)]}; \qquad w_\varepsilon(x) = \varphi(x)h_\varepsilon'(x) + \psi(x)[\alpha + \beta h_\varepsilon(x)];$$

and

$$q_\varepsilon(x) = \frac{w_\varepsilon(x)}{\psi(x)[\alpha + \beta h_\varepsilon(x)]} = \Delta_\varepsilon(x) + \frac{b}{\beta}$$

for $x \in (r_1, r_2) - \{\lambda\}$ and $0 < \varepsilon \ll 1$. Since $h_\varepsilon = h$ on $(r_1, x_2]$ and $0 < \varepsilon \ll 1$, one obtains that $w_\varepsilon(x) = w(x)$ and $\Delta_\varepsilon(x) = \frac{\varphi(x)h'(x)}{\psi(x)[\alpha+\beta h(x)]}$ if $x \in (r_1, x_2] - \{\lambda\}$ and $0 < \varepsilon \ll 1$. Thus, $\Delta_\varepsilon'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varphi(x)h'(x)}{\psi(x)[\alpha+\beta h(x)]}\right) > 0$ on $x \in (r_1, x_2] - \{\lambda\}$ and $0 < \varepsilon \ll 1$. If $x \in (x_2, r_2)$ and $0 < \varepsilon \ll 1$, then, since $h_\varepsilon(x) = h(x) + \varepsilon H(x)$ and (2.3), we have

$$\begin{aligned} w_\varepsilon(x) &= \varphi(x)(h'(x) + \varepsilon H'(x)) + \psi(x)[a + b(h(x) + \varepsilon H(x))] \\ &= w(x) + \varepsilon\varphi(x)\left[H'(x) + b\frac{\psi(x)}{\varphi(x)}H(x)\right] = (1 + \varepsilon\theta(x))w(x) \end{aligned}$$

and

$$\begin{aligned} q_\varepsilon(x) &= \frac{w_\varepsilon(x)}{\psi(x)[\alpha + \beta h(x) + \varepsilon\beta H(x)]} \\ &= (1 + \varepsilon\theta(x)) \cdot \frac{w(x)}{\psi(x)[\alpha + \beta h(x)]} \cdot \frac{\alpha + \beta h(x)}{\alpha + \beta h(x) + \varepsilon\beta H(x)}. \end{aligned}$$

Hence, from (2.2), (2.3) and (A6), we have

$$\begin{aligned} \Delta_\varepsilon'(x) &= q_\varepsilon'(x) \\ &= (1 + \varepsilon\theta(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w(x)}{\psi(x)[\alpha + \beta h(x)]}\right) \cdot \frac{\alpha + \beta h(x)}{\alpha + \beta h(x) + \varepsilon\beta H(x)} \\ &\quad + \varepsilon\theta'(x) \cdot \frac{w(x)}{\psi(x)[\alpha + \beta h(x)]} \cdot \frac{\alpha + \beta h(x)}{\alpha + \beta h(x) + \varepsilon\beta H(x)} \\ &\quad + (1 + \varepsilon\theta(x)) \cdot \frac{w(x)}{\psi(x)[\alpha + \beta h(x)]} \cdot \frac{\varepsilon\beta[\beta h'(x)H(x) - (\alpha + \beta h(x))H'(x)]}{[\alpha + \beta h(x) + \varepsilon\beta H(x)]^2} \\ &> \varepsilon\theta'(x) \cdot \frac{w(x)}{\psi(x)[\alpha + \beta h(x)]} \cdot \frac{\alpha + \beta h(x)}{\alpha + \beta h(x) + \varepsilon\beta H(x)} \\ &\quad + (1 + \varepsilon\theta(x)) \cdot \varepsilon\beta\frac{w(x)}{\psi(x)[\alpha + \beta h(x)]} \cdot \frac{w(x)[\beta H(x) - (\alpha + \beta h(x))\theta(x)]}{\varphi(x)[\alpha + \beta h(x) + \varepsilon\beta H(x)]^2} \\ &> \frac{\varepsilon w(x)}{\psi(x)[\alpha + \beta h(x) + \varepsilon\beta H(x)]} \end{aligned}$$

$$\times \left(\theta'(x) - (1 + \varepsilon\theta(x)) \frac{\theta(x)}{\varphi(x)} \frac{\beta w(x)}{\alpha + \beta h(x) + \varepsilon \beta H(x)} \right)$$

$$> \frac{\varepsilon w(x)}{\psi(x)[\alpha + \beta h(x) + \varepsilon \beta H(x)]} \left(\theta'(x) - (1 + \theta(x)) \frac{\theta(x)}{\varphi(x)} \frac{\beta w(x)}{\alpha + \beta h(x)} \right) = 0.$$

Hence, system (1.3) possesses at most one limit cycle, and if it exists it is stable. This completes the proof of theorem 2.2. □

Remark. If $\alpha = 1$ and $\beta = 0$ then theorem 2.2 is the criterion for the uniqueness of limit cycle given by Zhang [2].

Theorem 2.3. *Let assumptions (A1)–(A4) hold. Assume*

(i) There exist $\alpha, \beta \geqslant 0$ *such that* $\alpha + \beta h(x) > 0$ *and* $q(x) = \frac{\varphi(x)h'(x) - \varphi(\lambda)h'(\lambda)}{\psi(x)(\alpha + \beta h(x))}$ *is* C^1 *and* $q'(x) \geqslant 0$ *on* (r_1, r_2);

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}(\psi(x)(\alpha + \beta h(x))) > 0$ *on* (r_1, r_2).

Then system (1.3) possesses at most one limit cycle in Ω.

Proof. The proof will be divided into two cases. □

Case 1. $h'(\lambda) \geqslant 0$. Since $q \in C^1$ and $q'(x) \geqslant 0$ on (r_1, r_2), we have

$$\varphi(x)h'(x) - \varphi(\lambda)h'(\lambda) = q(x)\psi(x)(\alpha + \beta h(x)) \geqslant q(\lambda)\psi(x)(\alpha + \beta h(x)) \qquad \text{on} \quad (r_1, r_2).$$

Hence, $\varphi(x)h'(x) \geqslant q(\lambda)\psi(x)(\alpha + \beta h(x)) + \varphi(\lambda)h'(\lambda)$ on (r_1, r_2). Thus, theorem 2.1 implies that system (1.3) has no periodic orbit in Ω.

Case 2. $h'(\lambda) < 0$. Since $q(x) = \frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))} - \frac{\varphi(\lambda)h'(\lambda)}{\psi(x)(\alpha + \beta h(x))}$ on $(r_1, r_2) - \{\lambda\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\varphi(x)h'(x)}{\psi(x)(\alpha + \beta h(x))} \right) = q'(x) + \varphi(\lambda)h'(\lambda) \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1}{\psi(x)(\alpha + \beta h(x))} \right) > 0$$

on $(r_1, r_2) - \{\lambda\}$. Hence, the assertion follows from theorem 2.2.

Remark. If $e_\star$ is locally asymptotically stable, then condition (2) in theorem 2.3 is not necessary.

3. Examples

We present some models from the chemical literature.

Examples 3.1. The Brusselator [12, 13] is a simple model of a hypothetical chemical oscillator, named after the home of the scientists who proposed it. In dimensionless form, the system is

$$x'(t) = 1 - (b + 1)x + ax^2 y; \qquad y'(t) = bx - ax^2 y \tag{3.1}$$

where $a, b > 0$ are parameters and $x, y \geqslant 0$ are dimensionless concentrations.

Theorem 3.1. *System (3.1) possesses at most one limit cycle in* $\mathbb{R}^2_+$.

Proof. Consider the change of variables:

$$u = -1/ax; \qquad v = x + y.$$

Then, system (3.1) reduces to

$$\begin{aligned} u'(t) &= au^2 + (b + 1)u + (v + 1/au) \equiv v - h(u), \\ v'(t) &= (au + 1)/au \equiv -\psi(u) \end{aligned} \tag{3.2}$$

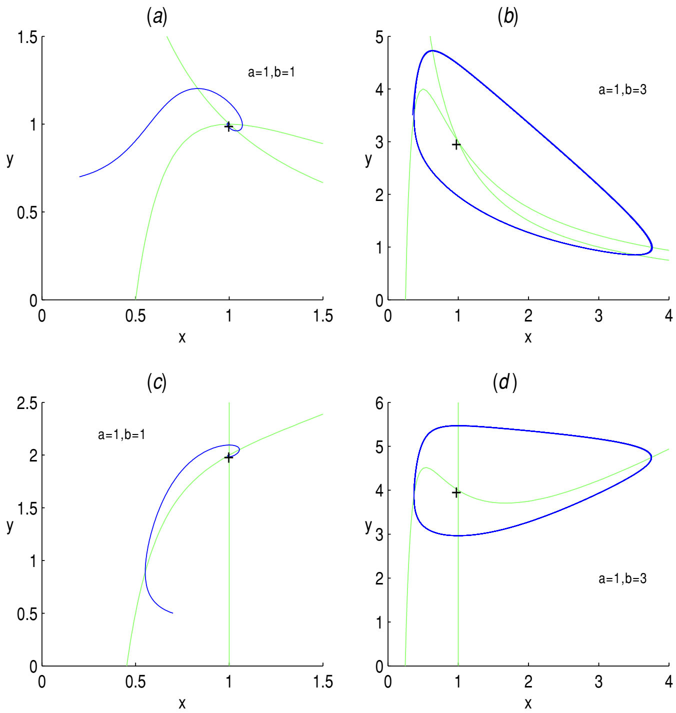

Figure 2. Parts (a), (c) illustrate the case when $e_\star$ is the global attractor of (3.1) and (3.2), respectively. Parts (b), (d) illustrate the case when a limit cycle is the global attractor of (3.1) and (3.2), respectively.

where $\varphi(u) = 1$, $\psi(u) = -(u-\lambda)/u$, $h(u) = -(a^2u^3 + a(b+1)u^2 + 1)/au$ and $\lambda = -1/a < 0$. The equilibrium of this system is $e_\star = (\lambda, h(\lambda))$. Since $h'(u) = -(2au + b + 1 - 1/au^2)$, we have

$$q(u) = (h'(u) - h'(\lambda))/\psi(u) = 2au + (\lambda + u)/a\lambda^2 u.$$

Hence, q is C^1 and $q'(u) = 2a - 1/a\lambda u^2 > 0$. Thus, the assertion follows from theorem 2.3 (see figure 2.) □

Example 3.2. To demonstrate oscillations in a simple chemical reaction, Schnackenberg [12, 14] provided the following hypothetical model:

$$\begin{array}{lcl} 2X + Y & \longleftrightarrow & 3X \\ A & \longrightarrow & Y \\ X & \longrightarrow & B. \end{array}$$

After nondimensionalizing, Schnackenberg obtained the system

$$x'(t) = a - x + x^2 y, \qquad y'(t) = b - x^2 y, \tag{3.3}$$

where $a, b > 0$ are parameters and $x, y > 0$ are dimensionless concentrations.

Theorem 3.2. *System (3.3) possesses at most one limit cycle in* $\mathbb{R}^2_+$.

Proof. Through the change of variables

$$u = -1/x, \qquad v = x + y,$$

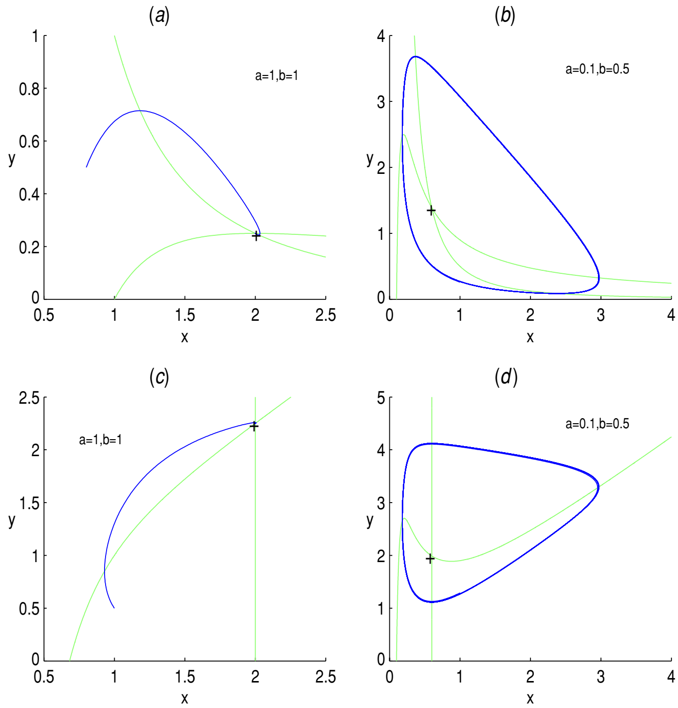

Figure 3. Parts (a), (c) illustrate the case when $e_\star$ is the global attractor of (3.3) and (3.4), respectively. Parts (b), (d) illustrate the case when a limit cycle is the global attractor of (3.3) and (3.4), respectively.

system (3.3) becomes

$$\begin{aligned} u'(t) &= v + au^2 + u + 1/u \equiv v - h(u), \\ v'(t) &= a + b + 1/u \equiv -\psi(u) \end{aligned} \tag{3.4}$$

where $\varphi(u) = 1$, $\psi(u) = 1/\lambda - 1/u$, $h(u) = -(au^2 + u + 1/u)$ and $\lambda = -(a+b)^{-1} < 0$. The positive equilibrium of this system is $e_\star = (\lambda, h(\lambda))$. The assertion is easy to verify by observing that

$$q(u) = (h'(u) - h'(\lambda))/\psi(u) = -2a\lambda u - (\lambda + u)/u\lambda.$$

Thus, q is $\mathcal{C}^1$ and $q'(u) = -2a\lambda + 1/u^2 > 0$. Hence, the assertion follows from theorem 2.3 (see figure 3.) □

Examples 3.3. Glycolysis [12, 15, 16] is one major reaction stage of the oxidation of glucose:

$$C_6H_{12}O_6 + 6O_2 \rightarrow 6CO_2 + 6H_2O + \text{energy},$$

and contains the following sequence of steps:

$$\text{glucose} \rightarrow \text{GGP} \rightarrow \text{FGP} \rightarrow \frac{\text{PFK}}{\text{ATP} \curvearrowright \text{ADP}} \rightarrow \text{FDP} \rightarrow \text{products},$$

where

$$\text{GGP} = \text{glucose-6-phosphate},$$
$$\text{FGP} = \text{fructose-6-phosphate},$$

FDP = fructose-1,6-diphosphate,

ATP = adenosine triphosphate,

ADP = adenosine diphosphate,

PFK = phosphofructokinase.

It is assumed that the enzyme phosphofructokinase has two states, one of which has a higher activity. ADP stimulates this allosteric regulatory enzyme and produces the more active form. Thus, a product of the reaction step mediated by PFK enhances the rate of reaction. A schematic version of the kinetics is

$$\text{substrates} \to \text{FGP} \overset{+}{\curvearrowleft} \text{ADP} \to \text{products}.$$

Equations for this system are as follows:

$$x'(t) = \delta - kx - xy^2, \qquad y'(t) = kx + xy^2 - y, \tag{3.5}$$

where $\delta, k > 0$ and x, y denote the concentrations for FGP, ADP, respectively.

Theorem 3.3. *System (3.5) possesses at most one limit cycle in* $\mathbb{R}_+^2$.

Proof. Consider the change of variables: $v = x + y$. Then, we have

$$\begin{aligned} y'(t) &= (y^2 + k)\left[v - \left(y + \frac{y}{y^2 + k}\right)\right] \equiv \varphi(y)[v - h(y)], \\ v'(t) &= \delta - y = -(y - \delta) \equiv -\psi(y), \end{aligned} \tag{3.6}$$

where $\varphi(y) = k + y^2$, $\psi(x) = y - \delta$, $h(y) = y + \frac{y}{y^2+k}$, and the equilibrium of this system is $e_\star = (\delta, h(\delta))$. Since $h'(y) = 1 + \frac{k-y^2}{(k+y^2)^2}$, we have $h'(y) \geqslant 0$ on $[0, \infty)$ if and only if $y^4 + (2k - 1)y^2 + k^2 + k \geqslant 0$ on $[0, \infty)$. This is equivalent to $k \geqslant 1/8$. Hence, according to theorem 2.1, (3.6) has no periodic solutions in $\mathbb{R}_+^2$. So, we always assume that $k \in (0, 1/8)$ in the following discussion.

Clearly, $0 < y_- = \sqrt{-k + \frac{1}{2}(1 - \sqrt{1 - 8k})} < y_+ = \sqrt{-k + \frac{1}{2}(1 + \sqrt{1 - 8k})}$ are the positive zeros of $h'(y) = 0$. Thus, $h'(y) > 0$ if $y \in [0, y_-) \cup (y_+, \infty)$ and $h'(y) < 0$ if $y \in (y_-, y_+)$. Hence, $e_\star$ is locally asymptotically stable or unstable if $\delta \in (0, y_-) \cup (y_+, \infty)$ or $\delta \in (y_-, y_+)$, respectively.

Now the proof is divided into the following three cases:

Case 1. $\delta \geqslant y_+$. If we can find $c \geqslant 0$ such that

$$\varphi(y)h'(y) \geqslant c\psi(y) \tag{3.7}$$

on $\mathbb{R}_+$, then according to theorem 2.1, we have that there are no periodic solutions on $\mathbb{R}_+^2$ for system (3.6). Clearly, (3.7) hold on $[0, y_-] \cup [y_+, \delta]$ for any $c \geqslant 0$. To make (3.7) hold on $(y_-, y_+) \cup (\delta, \infty)$, we let $Q(y) = \varphi(y)h'(y)/\psi(y)$ on $[0, \infty) - \{\delta\}$ and

$$L(y) = \frac{y^2 - y_-^2}{y^2 + k}(y + y_+).$$

Then,

$$Q(y) = L(y) + \frac{y^2 - y_-^2}{y^2 + k}\frac{y + y_+}{y - \delta}(\delta - y_+). \tag{3.8}$$

From (3.8), $\delta \geqslant y_+$ and $L(y)$ is an increasing function on (y_-, ∞), it follows that

$$Q(y) \geqslant L(y) \geqslant L(\delta) \qquad \text{on} \quad (\delta, \infty)$$

and

$$Q(y) \leqslant L(y) \leqslant L(y_+) \qquad \text{on} \quad (y_-, y_+).$$

Choose $c = L(y_+)$, then (3.7) hold on $\mathbb{R}_+$.

Case 2. $\delta \in (0, y_-]$. Similar to case 1, if we can find $c \leqslant 0$ such that

$$\varphi(y)h'(y) \geqslant c\psi(y)h(y) \tag{3.9}$$

on $\mathbb{R}_+$, then according to theorem 2.1, we have that there are no periodic solutions on $\mathbb{R}^2_+$ for system (3.6). Clearly, (3.9) hold on $[y_+, \infty)$ for any $c \leqslant 0$. To make (3.9) hold on $(0, \delta) \cup (\delta, y_+)$, we let $P(y) = \varphi(y)h'(y)/\psi(y)h(y)$ on $(0, \infty) - \{\delta\}$ and

$$l(y) = \frac{(y - y_+)(y + y_-)(y + y_+)}{y(y^2 + k + 1)}.$$

Then,

$$P(y) = l(y) + \frac{(\delta - y_-)(y - y_+)(y + y_-)(y + y_+)}{y(y - \delta)(y^2 + k + 1)}. \tag{3.10}$$

Since

$$\begin{aligned} l'(y)y^2(y^2 + k + 1)^2 &= y(y^2 + k + 1)\big(3y^2 + 2y_- y - y_+^2\big) \\ &\quad - (3y^2 + k + 1)\big(y^3 + y_- y^2 - y_+^2 y - y_- y_+^2\big) \\ &= -y_- y^4 + 2\big(k + 1 + y_+^2\big)y^3 + y_-\big(k + 1 + 3y_+^2\big)y^2 + (k + 1)y_- y_+^2 \\ &= 2\big(k + 1 + y_+^2\big)y^3 + y_- y^2\big(k + 1 + 3y_+^2 - y^2\big) + (k + 1)y_- y_+^2 \\ &> y_- y^2\big(y_+^2 - y^2\big) \geqslant 0 \qquad \text{on} \quad (0, y_+], \end{aligned}$$

we have $l(y)$ an increasing function on $(0, y_+)$. From (3.10) and $\delta \leqslant y_-$, it follows that

$$P(y) \geqslant l(y) \geqslant l(\delta) \qquad \text{on} \quad (\delta, y_+)$$

and

$$P(y) \leqslant l(y) \leqslant l(\delta) \qquad \text{on} \quad (0, \delta).$$

Choose $c = l(\delta)$, then (3.9) hold on $\mathbb{R}_+$.

Case 3. $y_- < \delta < y_+$. Let $\alpha = \delta(k + 1)/k$ and $\beta = 1$. Define

$$f(y) = \psi(y)(h(y) + \alpha) \qquad \text{on} \quad \mathbb{R}_+.$$

Clearly,

$$\begin{aligned} f'(y) &= \alpha + h(y) + \psi(y)h'(y) \\ &= \alpha + y + \frac{y}{y^2 + k} + (y - \delta)\frac{\big(y^2 - y_-^2\big)\big(y^2 - y_+^2\big)}{(y^2 + k)^2} && (3.11) \\ &= \alpha + y\frac{y^4 + 2ky^2 + k(k + 1)}{(y^2 + k)^2} - \delta\frac{\big(y^2 - y_-^2\big)\big(y^2 - y_+^2\big)}{(y^2 + k)^2} && (3.12) \\ &> \alpha + (y - \delta)\frac{\big(y^2 - y_-^2\big)\big(y^2 - y_+^2\big)}{(y^2 + k)^2} \\ &\equiv \alpha + (y - \delta)g(y). && (3.13) \end{aligned}$$

Since, $g(0) = (k+1)/k$ and $g(y) \geqslant 0$ is a decreasing function on $[0, y_-]$, we have $(y - \delta)g(y)$ is increasing function on $[0, y_-]$. Hence, $\alpha + (y - \delta)g(y) \geqslant \alpha - \delta(k + 1)/k = 0$ for all $y \in [0, y_-]$. Now from (3.11) and (3.12), we have $f'(y) > 0$ on $\mathbb{R}_+$. According to

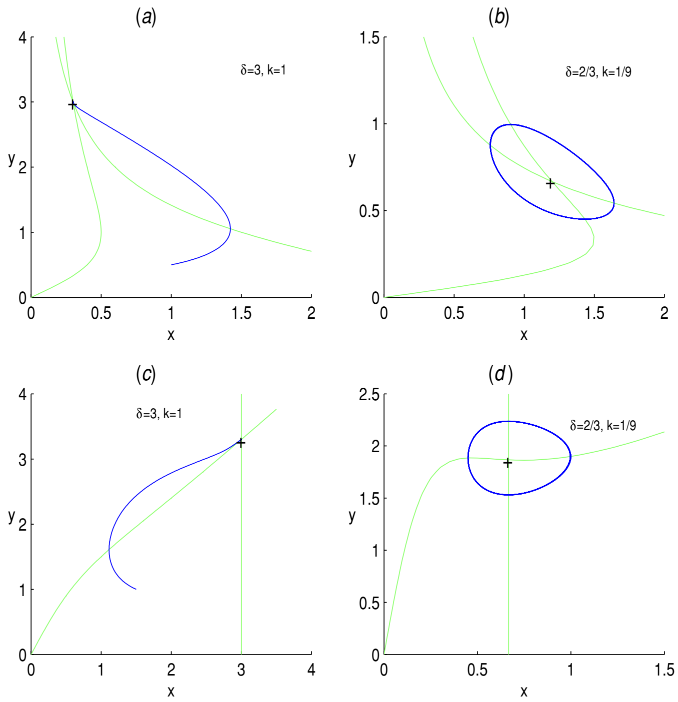

Figure 4. Parts (*a*), (*c*) illustrate the case when $e_\star$ is the global attractor of (3.5) and (3.6), respectively. Parts (*b*), (*d*) illustrate the case when a limit cycle is the global attractor of (3.5) and (3.6), respectively.

theorem 2.3, if we can prove $q(y) = (\varphi(y)h'(y) - \varphi(\delta)h'(\delta))/f(y)$ is a $\mathcal{C}^1$ function and $q'(y) \geqslant 0$ on $\mathbb{R}_+$. Then, since $h'(\delta) < 0$, we conclude that there is at most one limit cycle in $\mathbb{R}^2_+$. After a straightforward computation, one yields

$$\varphi(y)h'(y) - \varphi(\delta)h'(\delta) = \frac{y^2-\delta^2}{(\delta^2+k)(y^2+k)}[(\delta^2+k)y^2 + k\delta^2 + k(k-2)]$$

and

$$\begin{aligned} q(y) &= (\varphi(y)h'(y) - \varphi(\delta)h'(\delta))/f(y) \\ &= \frac{(y+\delta)(y^2+k\theta)}{y^3+\alpha y^2+(k+1)y+k\alpha} \end{aligned}$$

where $\theta = (\delta^2+k-2)/(\delta^2+k)$. Hence,

$$\begin{aligned} q'(y)(y^3+\alpha y^2+(k+1)y+k\alpha)^2 &= (3y^2+k\theta+2\delta y)(y^3+\alpha y^2+(k+1)y+k\alpha) \\ &\quad - (y^3+\delta y^2+k\theta y+k\delta\theta)(3y^2+2\alpha y+k+1) \\ &= (\alpha-\delta)y^4 + 2(k+1-k\theta)y^3 + (\delta(k+1)+3k\alpha-k\alpha\theta-3k\delta\theta)y^2 \\ &\quad + 2k\alpha\delta(1-\theta)y + k\theta(k\alpha-(k+1)\delta) \\ &= \frac{\delta}{k}y^4 + 2(k+1-k\theta)y^3 + (\delta(k+1)+3k\alpha-k\alpha\theta-3k\delta\theta)y^2 + 2k\alpha\delta(1-\theta)y. \end{aligned}$$

Since $\theta < 1$ and $0 < k < 1/8$, thus $q'(y) \geqslant 0$ on $[0, \infty)$. This completes the proof of example 3.3 (see figure 4). □

4. Concluding remark

Note that all the examples in section 3 take the form

$$W'(t) = p_1(W) + p_2(W)Z, \qquad Z'(t) = p_3(W) - p_2(W)Z, \tag{4.1}$$

with suitable functions p_1, p_2, p_3. Through the change of variables

$$x = W \qquad \text{and} \qquad y = W + Z,$$

system (4.1) becomes

$$x'(t) = p_2(x)y - xp_2(x) + p_1(x), \qquad y'(t) = p_1(x) + p_3(x), \tag{4.2}$$

which is a generalized Liénard system. According to theorem 2.2, if one can prove the monotonicity of the function

$$-\frac{p_2(x)(xp_2'(x) + p_2(x) - p_1'(x))}{(p_1(x) + p_3(x))(\alpha + \beta(xp_2(x) - p_1(x)))}$$

for suitable $\alpha, \beta \geqslant 0$, then system (4.1) has at most one limit cycle.

It should be pointed that although we have established theorem 2.2, the problem about the number of limit cycles for system (1.3) is still open.

Acknowledgments

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